

# METHOD OF MOMENTS ESTIMATION OF ORNSTEIN-UHLENBECK PROCESSES DRIVEN BY GENERAL LÉVY PROCESS

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ABSTRACT. Ornstein-Uhlenbeck processes driven by general Lévy process are considered in this paper. We derive strongly consistent estimators for the moments of the underlying Lévy process and for the mean reverting parameter of a discretely observed Lévy driven Ornstein-Uhlenbeck process. Moreover, we prove that the estimators are asymptotically normal. We use ergodicity arguments. Finally, we test the empirical performance of our estimators in a simulation study and we fit the model to real VIX data.

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## 1. INTRODUCTION

Given a positive number  $\lambda$  and a time-homogeneous Lévy process  $L$ , the Ornstein-Uhlenbeck (OU) process driven by  $L$  is defined by

$$(1) \quad Y_t = e^{-\lambda t} Y_0 + e^{-\lambda t} \int_0^{\lambda t} e^s dL_s,$$

where  $Y_0$  is assumed to be independent of  $\{L_t\}_{t \geq 0}$ . Following the terminology introduced by Barndorff-Nielsen and Shephard in [2], we shall call  $L$  the background driving Lévy process (BDLP). It is easy to see that (1) is the unique strong solution of the stochastic differential equation

$$(2) \quad dY_t = -\lambda Y_t dt + dL_{\lambda t}.$$

Under some regularity conditions on the Lévy measure of  $L$  and if  $\lambda > 0$ ,  $Y$  admits a unique invariant distribution  $F_Y$ . Owing to the scaling of the time index of  $L$  in (2) by  $\lambda$  (i.e. the term  $L_{\lambda t}$ ),  $F_Y$  is independent of  $\lambda$ .

Let us suppose now that we have discrete-time observations  $Y_0, Y_h, \dots, Y_{(n-1)h}$  with  $h > 0$  from  $\{Y_t\}_{t \geq 0}$  as it is defined by (1). The objective here is to estimate the parameters of the model using these discrete-time observations. In particular, we are interested in estimating  $\lambda$  and moments of  $L_1$ . We derive strongly consistent method of moments estimators and prove that they are asymptotically normal. In this paper, we consider  $\lambda$  and only the first two moments of  $L_1$ , i.e.  $\mathbb{E}L_1$  and  $\mathbb{E}L_1^2$ . However, the methodology can be extended to higher moments as well (see Remark 3.2). Similar methods of moments estimators have been used elsewhere as well (see Valdivieso et al [25] for example), but without theoretical justification. The proof

of their consistency and asymptotic normality is presented, according to the best knowledge of the author, for the first time in the present work.

Our motivation for studying this problem comes from continuous stochastic volatility models in financial mathematics. Barndorff-Nielsen and Shephard (in [2]; see also [3]) model stock price as a geometric Brownian motion and the diffusion coefficient of this motion as an OU process that is driven by a subordinator (a Lévy process that is nonnegative and nondecreasing). Other continuous stochastic volatility models can be found in Klüppelberg et al [15] and in Shephard [24]. Some papers that consider statistical inference of these models are Barndorff-Nielsen and Shephard [2], Brockwell et al [5], Haug et al [12], Jongbloed et al [13].

The paper is organized as follows. Section 2 presents several known results concerning OU processes and Lévy processes. In section 3 we consider strongly consistent estimators of the first two moments of  $L_1$  and of  $\lambda$  and we provide a methodology to express any moment of the stationary distribution of  $\{Y\}_{t \geq 0}$  in terms of the moments of  $L_1$ . In section 4, we prove that these estimators are asymptotically normal. Section 5 discusses modeling issues and simulation techniques and presents simulation results for gamma OU process and inverse Gaussian OU process. In section 6 we fit the model to real log(VIX) data and we argue that an OU model is a good candidate for modeling log(VIX). Finally, section 7 contains a summary and a discussion on future work.

We would like to mention here, that after completion of this work, the author learned about the results in Jongbloed et al [13]. In [13], the authors assume that  $L$  is a subordinator. Let  $F_L$  denote the Lévy measure of  $L$ ,  $Y$  the unique stationary solution to (1) (which exists if  $\int_{x>1} \log(x) F_L(dx) < \infty$  for example) and  $F_Y$  its probability law. The characteristic function of  $Y$  is given by

$$\phi_{F_Y}(t) := \int e^{itx} F_Y(dx) = \exp\left(\int_0^\infty [e^{itx} - 1] \frac{\kappa(x)}{x} dx\right),$$

where  $\kappa(x) = F_L(x, \infty)$ . Hence, the stationary distribution  $F_Y$ , of the OU process  $Y$ , is being determined by the canonical function  $\kappa(x)$ . In [13], the authors develop a nonparametric inference procedure for  $\lambda$  and for the canonical function  $\kappa(x)$ . The results in the present paper complement the results of [13].

## 2. ASSUMPTIONS AND PRELIMINARY RESULTS

Consider a probability space  $(\Omega, \mathfrak{F}, P)$  equipped with a filtration  $\mathfrak{F}_t$ .

**Definition 2.1.** *A one dimensional  $\mathfrak{F}_t$  adapted Lévy process is usually denoted by  $L_t = L_t(\omega)$ ,  $t \geq 0$ ,  $\omega \in \Omega$  and is a stochastic process that satisfies the following:*

- (i)  $L_t \in \mathfrak{F}_t$  for all  $t \geq 0$ .
- (ii)  $L_0 = 0$  a.s.
- (iii)  $L_t - L_s$  is independent of  $\mathfrak{F}_s$  and has the same distribution as  $L_{t-s}$ .
- (iv) It is a process continuous in probability.

We assume that we are working with a càdlàg Lévy process (i.e. it is right continuous with left limits). It is well known that every Lévy process has such a modification.

Furthermore, if  $F_L$  denotes the Lévy measure of  $L_1$ , we will assume that there exist a constant  $M > 0$  such that

$$(3) \quad \int_{|x|>1} e^{vx} F_L(dx) < \infty, \quad \text{for every } |v| \leq M.$$

Condition (3) guarantees that the moment generating function  $v \rightarrow \mathbb{E}e^{vL_1}$  exists at least for  $|v| \leq M$  (see Wolfe [27] and Eberlein and Raible [9]).

We shall write

- (i)  $\mathbb{E}L_1 = \mu$ .
- (ii)  $\text{Var}(L_1) = \sigma^2$ .

Moreover, we shall assume that  $Y_0$  is independent of  $\{L_t\}_{t \geq 0}$  and that

$$(4) \quad Y_0 \stackrel{\mathcal{D}}{=} \int_0^\infty e^{-s} dL_s.$$

The integral on the right hand side of (4) is well defined (see Sato [22] for example). The following proposition, which is a reformulation of Propositions 1 and 2 in Brockwell [6], characterizes the stationarity of the OU process  $\{Y_t\}_{t \geq 0}$ .

**Proposition 2.2.** *If  $Y_0$  is independent of  $\{L_t\}_{t \geq 0}$  and  $\mathbb{E}L_1^2 < \infty$  then  $\{Y_t\}_{t \geq 0}$  is weakly stationary if and only if  $\lambda > 0$  and  $Y_0$  has the same mean and variance as  $\int_0^\infty e^{-s} dL_s$ . If in addition  $Y_0$  has the same distribution as  $\int_0^\infty e^{-s} dL_s$ , then  $\{Y_t\}_{t \geq 0}$  is strictly stationary and vice-versa.*

In Masuda [19] now, the author proves, under mild regularity conditions, that the OU process  $Y$  is strong Feller, its probability law has a smooth transition density, is ergodic and exponentially  $\beta$ -mixing (strong mixing). Before mentioning the results of [19] that we will use in the present paper, let us recall the definitions of a self-decomposable law on  $\mathbb{R}$  and of  $\beta$ -mixing.

**Definition 2.3.** *Let  $\lambda$  be a positive number. Then, an infinitely divisible distribution  $F_Y$  is called  $\lambda$ -self-decomposable, if there exists a random variable  $X = X_{t,\lambda}$ , such that, for each  $t \in \mathbb{R}_+$*

$$\phi_{F_Y}(u) = \phi_{F_Y}(e^{-\lambda t}u)\phi_{F_X}(u), u \in \mathbb{R},$$

where  $\phi_{F_Y}(u)$  and  $\phi_{F_X}(u)$  are the characteristic functions corresponding to  $F_Y$  and  $F_X$  respectively. For the sake of notational convenience we will just say that  $F_Y$  is called self-decomposable.

If  $\int_{|x|>1} \log(|x|)F_L(dx) < \infty$ , then the class of all possible invariant distributions of  $Y$  forms the class of all self-decomposable distributions  $F_Y$  (see Sato [22]). In particular, the latter is implied by (3).

**Definition 2.4.** *For a stationary process  $Y = \{Y_t\}_{t \geq 0}$  define the  $\sigma$ -algebras  $\mathfrak{F}_1 = \mathfrak{F}_{(0,u)} = \sigma(\{Y_v\}, 0 \leq v < u)$  and  $\mathfrak{F}_2 = \mathfrak{F}_{[u+t,\infty)} = \sigma(\{Y_v\}, v \geq u+t)$ . Then*

- (i)  *$Y$  is called  $\beta$ -mixing (or strong mixing) if:*

$$\beta(t) = \sup_{A \in \mathfrak{F}_1, B \in \mathfrak{F}_2} |P(A \cap B) - P(A)P(B)| \rightarrow 0 \text{ as } t \rightarrow \infty.$$

- (ii)  *$Y$  is called  $\beta$ -mixing with exponential rate if for some  $k > 0$  and  $a > 0$ :*

$$\beta(t) \leq ke^{-at} \text{ for } t \geq 0.$$

The following theorem is Theorem 4.3 in Masuda [19] and discusses the mixing properties of  $\{Y_t\}_{t \geq 0}$ .

**Theorem 2.5.** *Let  $\lambda > 0$  and  $\{Y_t\}_{t \geq 0}$  be the strictly stationary OU process given by (1) with self-decomposable marginal distribution  $F_Y$ . If we have that*

$$\int_{\mathbb{R}} |x|^p F_Y dx < \infty$$

for some  $p > 0$ , then there exists a constant  $a > 0$  such that  $\beta(t) = O(e^{-at})$  as  $t \rightarrow \infty$ . In particular,  $Y$  is ergodic.

### 3. METHOD OF MOMENTS ESTIMATION

We aim at estimation of the model parameters  $\theta_0 = (\mu, \sigma^2, \lambda)$  from a sample of equally spaced observations from (1) by matching moments and empirical autocorrelation function to their theoretical counterparts.

Proposition 3.1 below relates the theoretical moments of  $L_1$  with the theoretical moments of the stationary distribution  $F_Y$  of  $\{Y_t\}$ .

**Proposition 3.1.** *Suppose that  $\{L_t\}_{t \geq 0}$  is a Lévy process such that  $\mathbb{E}L_1 = \mu < \infty$ ,  $\text{Var}L_1 = \sigma^2 < \infty$  and that (3) holds. Let  $M$  be the largest constant satisfying (3) and assume that  $\lambda < M$ . Then, the following are true*

- (i)  $\mathbb{E}Y_0 = \mu$
- (ii)  $\text{Var}Y_0 = \frac{\sigma^2}{2}$

*Proof.* Let  $\gamma(v)$  be the cumulant function of  $L_1$ , i.e.

$$(5) \quad \gamma(v) = \ln \mathbb{E}e^{vL_1}$$

By the Lévy- Khinchine representation Theorem we get that  $\gamma(v)$  has the form

$$(6) \quad \gamma(v) = bv + \frac{c}{2}v^2 + \int_{\mathbb{R}} (e^{vx} - 1 - vx)F_L(dx),$$

which is valid for  $|v| \leq M$ . Moreover,  $\gamma$  is continuously differentiable (see Lukacs [17]).

Using the assumptions  $\mathbb{E}L_1 = \mu$  and  $\text{Var}L_1 = \sigma^2$  and relations (5) and (6), it is easy to see that  $b = \mu$  and  $c = \sigma^2 - \int_{\mathbb{R}} x^2 F_L(dx)$ .

In order to calculate  $\mathbb{E}Y_0$  and  $\text{Var}Y_0$  we use the following formula:

$$(7) \quad \mathbb{E}e^{\int_0^\infty \lambda e^{-s} dL_s} = e^{\int_0^\infty \gamma(\lambda e^{-s}) ds},$$

which is valid since  $\lambda < M$  (see Lemma 3.1 of Eberlein and Raible [9]).

Recall now that we have assumed  $Y_0 = \int_0^\infty e^{-s} dL_s$  in distribution. The latter and (7) imply that:

$$(8) \quad \begin{aligned} \mathbb{E}Y_0 &= \frac{d}{d\lambda} \mathbb{E}e^{\int_0^\infty \lambda e^{-s} dL_s} \Big|_{\lambda=0} = \\ &= \frac{d}{d\lambda} e^{\int_0^\infty \gamma(\lambda e^{-s}) ds} \Big|_{\lambda=0} = \\ &= \mu \end{aligned}$$

In a similar way we get that  $\mathbb{E}Y_0^2 = \frac{\sigma^2}{2} + \mu^2$ . This concludes the proof of the proposition.  $\square$

**Remark 3.2.** *We would like to note here, that the proof of Proposition 3.1 can be used for the calculation of higher moments of  $Y_0$ .*

It follows directly by (1) that the theoretical autocovariance and autocorrelation function of  $Y_t$  are given by the formulas

- (i) autocovariance:  $\gamma(h) = \text{cov}(Y_{t+h}, Y_t) = \frac{\sigma^2}{2} e^{-\lambda h}$ , for  $h \in \mathbb{N}_0$ .
- (ii) autocorrelation:  $\rho(h) = \text{corr}(Y_{t+h}, Y_t) = e^{-\lambda h}$ , for  $h \in \mathbb{N}_0$ .

On the other hand, the empirical moments, autocorrelation and autocovariance function are given by the formulas below. Let  $d \geq 0$  be fixed. Then, we have:

- (i) Sample mean:  $\bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i$ .
- (ii) Sample variance:  $\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})^2$ .
- (iii) Sample autocovariance:  $\hat{\gamma}_n = (\hat{\gamma}_n(0), \hat{\gamma}_n(1), \dots, \hat{\gamma}_n(d))^T$  where for  $h \in \{0, \dots, d\}$  we define  $\hat{\gamma}_n(h) = \frac{1}{n} \sum_{i=1}^{n-h} (Y_{i+h} - \bar{Y})(Y_i - \bar{Y})$ .
- (iv) Sample autocorrelation:  $\hat{\rho}_n = (\hat{\rho}_n(0), \hat{\rho}_n(1), \dots, \hat{\rho}_n(d))^T$  where for  $h \in \{0, \dots, d\}$  we define  $\hat{\rho}_n(h) = \frac{\hat{\gamma}_n(h)}{\hat{\gamma}_n(0)}$ .

We have the following Theorem:

**Theorem 3.3.** *Let  $\mu, \sigma^2, \gamma(\cdot), \hat{\gamma}_n(\cdot), \rho(\cdot)$  and  $\hat{\rho}_n(\cdot)$  be defined as above. Then, the following statements are true*

- (i)  $\bar{Y} \xrightarrow{n \rightarrow \infty} \mu$  almost surely
- (ii)  $\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})^2 \xrightarrow{n \rightarrow \infty} \frac{\sigma^2}{2}$  almost surely
- (iii)  $(\hat{\gamma}_n(1), \dots, \hat{\gamma}_n(d)) \xrightarrow{n \rightarrow \infty} (\gamma(1), \dots, \gamma(d))$  almost surely
- (iv)  $(\hat{\rho}_n(1), \dots, \hat{\rho}_n(d)) \xrightarrow{n \rightarrow \infty} (\rho(1), \dots, \rho(d))$  almost surely

*Proof.* Due to our assumptions, the process  $\{Y_t\}_{t \geq 0}$  is strictly stationary. Moreover by Theorem 2.5 it is also  $\beta$ -mixing with exponential decaying rate. These two results imply ergodicity of  $\{Y_t\}_{t \geq 0}$ . The latter together with strict stationarity imply that empirical moments and sample autocovariance functions are strongly consistent estimators of the corresponding theoretical quantities Billingsley [4]. Then, the statement of the Theorem follows.  $\square$

For the mean reverting parameter  $\lambda$  we have the following Lemma.

**Lemma 3.4.** *Let  $K$  be a compact subset of  $\mathbb{R}_+$  such that the true value of  $\lambda$ , say  $\lambda_o$ , belongs to  $K$  and let  $\hat{\lambda}_n = \operatorname{argmin}_{\lambda \in K} \sum_{h=1}^d (\hat{\rho}_n(h) - e^{-\lambda h})^2$ . Then  $\hat{\lambda}_n$  exists, is locally unique and*

$$(9) \quad \hat{\lambda}_n \xrightarrow{n \rightarrow \infty} \lambda_o \text{ almost surely.}$$

*Proof.* Consider the functions  $\Delta_n(\lambda) = \sum_{h=1}^d (\hat{\rho}_n(h) - \rho_\lambda(h))^2$  and  $\Delta_0(\lambda) = \sum_{h=1}^d (\rho_{\lambda_o}(h) - \rho_\lambda(h))^2$  where  $\rho_\lambda(h) = e^{-\lambda h}$ . Theorem 3.3 implies that for all  $\lambda \in K$ :

$$\Delta_n(\lambda) \xrightarrow{n \rightarrow \infty} \Delta_0(\lambda) \text{ almost surely.}$$

By Theorem II.1 in Andresen and Grill [1] we have

$$\sup_{\lambda \in K} |\Delta_n(\lambda) - \Delta_0(\lambda)| \xrightarrow{n \rightarrow \infty} 0 \text{ almost surely.}$$

Observe now that  $\Delta_0(\lambda)$  is a sum of nonnegative terms. It becomes zero if and only if  $\lambda = \lambda_o$ . Hence,  $\Delta_0(\lambda)$  has a unique minimum at  $\lambda = \lambda_o$  which is equal to zero. We get

$$\Delta_n(\lambda_o) \xrightarrow{n \rightarrow \infty} 0 \text{ almost surely.}$$

Furthermore, for  $n$  finite we have that  $0 \leq \Delta_n(\hat{\lambda}_n) \leq \Delta_n(\lambda_o)$ . Therefore we get

$$\Delta_n(\hat{\lambda}_n) \xrightarrow{n \rightarrow \infty} 0 \text{ almost surely.}$$

Moreover, we have

$$\begin{aligned}
|\Delta_n(\hat{\lambda}_n) - \Delta_0(\hat{\lambda}_n)| &= \left| \sum_{h=1}^d [\hat{\rho}_n^2(h) - \rho_{\lambda_0}^2(h) + 2\rho_{\hat{\lambda}_n}(h)(\rho_{\lambda_0}(h) - \hat{\rho}_n(h))] \right| \leq \\
&\leq \sum_{h=1}^d [|\hat{\rho}_n(h)| + |\rho_{\lambda_0}(h)| + 2|\rho_{\hat{\lambda}_n}(h)|] |\rho_{\lambda_0}(h) - \hat{\rho}_n(h)| \leq \\
&\leq 4 \sum_{h=1}^d |\rho_{\lambda_0}(h) - \hat{\rho}_n(h)| \xrightarrow{n \rightarrow \infty} 0 \text{ almost surely.}
\end{aligned}$$

Here, we used the relation  $|\hat{\rho}_n(h)| \leq 1$  which follows immediately from Cauchy-Schwarz inequality. The above imply that

$$\Delta_0(\hat{\lambda}_n) \xrightarrow{n \rightarrow \infty} 0 \text{ almost surely.}$$

But,  $\lambda_0$  is the unique minimum of  $\Delta_0(\lambda)$  and it satisfies  $\Delta_0(\lambda_0) = 0$ . Thus,

$$\Delta_0(\hat{\lambda}_n) \xrightarrow{n \rightarrow \infty} \Delta_0(\lambda_0) = 0 \text{ almost surely.}$$

Hence, we easily conclude (Corollary II.2 in [1]) that  $\hat{\lambda}_n$  is locally uniquely determined and that

$$\hat{\lambda}_n \xrightarrow{n \rightarrow \infty} \lambda_0 \text{ almost surely.}$$

□

**Remark 3.5.** *Theorem 3.3 and Lemma 3.4 give us two strongly consistent estimators for  $\lambda$ . The first one is  $\hat{\lambda}_{1,n} = -\log(\hat{\rho}_n(1))$  and the second one is  $\hat{\lambda}_{2,n} = \operatorname{argmin}_{\lambda} \sum_{h=1}^d (\hat{\rho}_n(h) - e^{-\lambda h})^2$ . One could use, for example,  $\hat{\lambda}_{1,n}$  as an initial value to an algorithm that calculates  $\hat{\lambda}_{2,n}$ .*

Summarizing, we have that  $\hat{\mu}_n$ ,  $\hat{\sigma}_n^2$  and  $\hat{\lambda}_{1,n}$ ,  $\hat{\lambda}_{2,n}$  are strongly consistent estimators of  $\mu$ ,  $\sigma^2$  and  $\lambda$  respectively, where:

$$\begin{aligned}
(10) \quad \hat{\mu}_n &= \frac{1}{n} \sum_{i=1}^n Y_i \\
\hat{\sigma}_n^2 &= 2 \frac{1}{n} \sum_{i=1}^n (Y_i - \hat{\mu}_n)^2 \\
\hat{\lambda}_{1,n} &= -\log(\hat{\rho}_n(1)) \\
\hat{\lambda}_{2,n} &= \operatorname{argmin}_{\lambda} \sum_{h=1}^d (\hat{\rho}_n(h) - e^{-\lambda h})^2.
\end{aligned}$$

**Remark 3.6.** *For a stationary model, the parameter  $\lambda$  has to be positive. However, if we compute  $\hat{\lambda}_{2,n}$  as the unrestricted minimum  $\hat{\lambda}_{2,n} = \operatorname{argmin}_{\lambda \in \mathbb{R}_+} \sum_{h=1}^d (\hat{\rho}_n(h) - e^{-\lambda h})^2$  we may end up with a negative estimator  $\hat{\lambda}_n$ . In this case, we define the estimator of  $\lambda$  to be zero and we take this as an indication that the data is not stationary.*

## 4. ASYMPTOTIC PROPERTIES OF THE MOMENT ESTIMATORS

In this section we prove that the estimators defined by (10) are asymptotically normal.

If  $\beta$  is a vector, then we define by  $\beta^T$  its transpose. We begin with the following central limit theorem.

**Theorem 4.1.** *Let us assume that there exists a  $\delta > 0$  such that  $\mathbb{E}Y_0^{4+\delta} < \infty$ . Define*

$$\begin{aligned}\hat{\psi}_n &= (\hat{\mu}_n, \hat{\gamma}_n(0), \hat{\gamma}_n(1), \dots, \hat{\gamma}_n(d))^T \\ \psi_o &= (\mu, \gamma(0), \gamma(1), \dots, \gamma(d))^T \\ \Sigma &= [\sigma_{k,l}]_{k,l=1}^{d+2} \text{ with elements} \\ \sigma_{k,l} &= \text{cov}(Z_1^k, Z_1^l) + 2 \sum_{i=1}^{\infty} \text{cov}(Z_1^k, Z_{i+1}^l) \text{ where} \\ Z_i &= (Y_i, (Y_i - \mu)^2, (Y_{i+1} - \mu)(Y_i - \mu), \dots, (Y_{i+d} - \mu)(Y_i - \mu))^T\end{aligned}$$

Then, the following holds:

$$(11) \quad \sqrt{n}(\hat{\psi}_n - \psi_o) \xrightarrow{\mathfrak{D}} N(0, \Sigma)$$

where  $N(0, \Sigma)$  is the multivariate normal distribution with mean 0 and variance-covariance matrix  $\Sigma$ .

*Proof.* The proof of this theorem is similar to the proof of Proposition 3.7 of Haug et al [12]. Let us define

$$\begin{aligned}\text{(i)} \quad \gamma_n^*(h) &= \frac{1}{n} \sum_{i=1}^n (Y_{i+h} - \mu)(Y_i - \mu), h \in \{0, \dots, d\}. \\ \text{(ii)} \quad \gamma_n^* &= (\gamma_n^*(0), \dots, \gamma_n^*(d))^T.\end{aligned}$$

We first prove that (11) is true with  $\hat{\psi}_n^* = (\hat{\mu}_n, \hat{\gamma}_n^*(0), \hat{\gamma}_n^*(1), \dots, \hat{\gamma}_n^*(d))^T$  in place of  $\hat{\psi}_n$ .

By the well known Cramer-Wold device, it is sufficient to prove that for every  $\beta \in \mathbb{R}^{d+2}$  such that  $\beta^T \Sigma \beta > 0$  we have

$$(12) \quad \sqrt{n} \left( \frac{1}{n} \sum_{i=1}^n \beta^T Z_i - \beta^T \psi_o \right) \xrightarrow{\mathfrak{D}} N(0, \beta^T \Sigma \beta).$$

It is well known (see [4] for example) that strong mixing and the corresponding decaying rate are preserved under linear transformations. Thus, the sequence  $\{\beta^T Z_i\}$  is strong mixing with exponential decaying rate. Since, by assumption  $\mathbb{E}|Z_1|^{2+\epsilon}$  for some  $\epsilon > 0$ , the central limit theorem for strong mixing processes is applicable (Theorem 7.3.1 in Ethier and Kurtz [10]). Hence, we have as  $n \rightarrow \infty$  that

$$(13) \quad \sqrt{n} \left( \frac{1}{n} \sum_{i=1}^n \beta^T Z_i - \beta^T \psi_o \right) \xrightarrow{\mathfrak{D}} N(0, \tilde{\sigma}^2).$$

But, we easily see that  $\tilde{\sigma}^2 = \text{var}(\beta^T Z_1) + 2 \sum_{i=1}^{\infty} \text{cov}(\beta^T Z_1, \beta^T Z_{i+1}) = \beta^T \Sigma \beta$ . So (12) holds.

Now recall that by Theorem 3.3 we have

$$(14) \quad \hat{\psi}_n \xrightarrow{n \rightarrow \infty} \psi_o \text{ almost surely.}$$

Following the proof of proposition 7.3.4 of Brockwell and Davis [7] we get

$$(15) \quad \sqrt{n} \left( \frac{1}{n} \sum_{i=1}^n \beta^T Z_i - \beta^T \hat{\psi}_n \right) \xrightarrow{n \rightarrow \infty} 0 \text{ in probability.}$$

Therefore,  $\hat{\psi}_n$  has the same asymptotic behavior as  $\hat{\psi}_n^*$ . The latter and (12) imply the Theorem.  $\square$

**Corollary 4.2.** *Let the conditions of Theorem 4.1 hold. Then we have*

$$\sqrt{n}(\hat{\rho}_n - \rho) \xrightarrow{\mathcal{D}} N(0, \Sigma_\rho).$$

*Proof.* It follows directly by Theorem 4.1 and delta method (Theorem 3.1 in A.W.van der Vaart [26]).  $\square$

Finally, we prove central limit theorem for  $\hat{\theta}_n = (\hat{\mu}_n, \hat{\sigma}_n^2, \hat{\lambda}_{2,n})^T$ . Let us denote  $\sigma_Y^2 = \gamma(0) = \text{Var}(Y_0) = \frac{\sigma^2}{2}$  and define the following mappings.

$$(16) \quad G : \mathbb{R} \times [0, \infty)^2 \longrightarrow \mathbb{R} \times [0, \infty)^2 : G(\mu, \sigma_Y^2, \lambda) = \begin{cases} (\mu, 2\sigma_Y^2, \lambda), & \lambda > 0 \\ (\mu, 2\sigma_Y^2, 0), & \lambda \leq 0. \end{cases}$$

$$(17) \quad F : \mathbb{R}_+^{d+1} \longrightarrow \mathbb{R}_+ : F(\hat{\rho}) = \underset{h=0}{\text{argmin}}_\lambda \sum_{h=0}^d (\hat{\rho}_n(h) - e^{-\lambda h})^2 = \hat{\lambda}_{2,n}$$

and  $H$  as follows:

$$(18) \quad H : \mathbb{R}^{d+2} \longrightarrow \mathbb{R} \times [0, \infty)^2 : H(\mu, \gamma^T) = G(\mu, \sigma_Y^2, F(\rho)),$$

where  $\rho(h) = \frac{\gamma(h)}{\gamma(0)}$  for  $h = 0, \dots, d$ .

**Theorem 4.3.** *Let the conditions of Theorem 4.1 be satisfied. Let us define:*

$$\begin{aligned} \hat{\theta}_n &= (\hat{\mu}_n, \hat{\sigma}_n^2, \hat{\lambda}_{2,n})^T \\ \theta_o &= (\mu, \sigma^2, \lambda)^T. \end{aligned}$$

*Then the following holds:*

$$(19) \quad \sqrt{n}(\hat{\theta}_n - \theta_o) \xrightarrow{\mathcal{D}} \left[ \frac{\partial H(\mu, \gamma^T)}{\partial(\mu, \gamma^T)} \right] N(0, \Sigma)$$

*Proof.* It follows directly by Theorem 4.1 and delta method applied to the differentiable map  $H$ .  $\square$

## 5. MODELING AND SIMULATION

In this section we discuss modeling issues of Lévy driven OU processes and present some simulation results for a gamma OU process and an inverse Gaussian OU process. We use the simulated data to test the performance of our estimators. However, first we mention the necessary ingredients for the simulation process.

A very important ingredient in modeling of Lévy driven OU processes is the connection between the Lévy density of the stationary distribution of  $Y$  to the Lévy density of the probability law of  $L_1$ . In particular we have the following proposition.



**Proposition 5.1.** *Assume that the Lévy density of  $Y$ ,  $\nu_Y(x)$ , is differentiable and denote the Lévy density of the probability law of  $L_1$  by  $\nu_L(x)$ . Then the following relation holds.*

$$(20) \quad \nu_L(x) = -\nu_Y(x) - x\nu_Y'(x).$$

*Proof.* It follows directly by the fact that the stationary solution,  $Y$ , to (2) satisfies

$$Y \stackrel{\mathcal{D}}{=} \int_0^\infty e^{-\lambda s} dL(\lambda s).$$

See [2] and [3] for more details.  $\square$

Hence, given  $\nu_L(x)$  we can find  $\nu_Y(x)$  and vice-versa. One can specify the law of the one dimensional marginal distribution of the OU process  $Y$  and work out the density of the BDLP,  $L_1$ . One can also go the other way and model through the BDLP. Of course, there are constraints on valid BDLP's which must be satisfied. In particular, if

$$\int_{\mathbb{R}} \min\{1, x^2\} \nu_L(x) dx$$

then  $\nu_L(x)$  is the density of a Lévy jump process  $L$  and there exists an OU process  $Y$  such that  $L$  is the BDLP of  $Y$ . A very good survey on the relation between several distributions of  $Y$  and  $L$  is Barndorff-Nielsen and Shephard [3] (see also Schoutens [23]).

Another important ingredient in simulations is the infinite series representation of Lévy integrals (see Rosinski [21]). For simplicity, we restrict attention to Lévy processes,  $L$ , that are subordinators, i.e. they are nonnegative and nondecreasing. It is easy to see that subordinators have no Gaussian component, nonnegative drift and a Lévy measure that is zero on the negative half-line. If  $Y$  models stochastic volatility then it has to be positive and such a choice of the BDLP guarantees that.

Let us denote by  $\Gamma_L^\perp$  the tail mass function of  $\nu_L$ , i.e.

$$(21) \quad \Gamma_L^+(x) = \int_x^\infty \nu_L(y) dy$$

and by  $\Gamma_L^{-1}$  the generalized inverse function of  $\Gamma_L^+$ , i.e.

$$(22) \quad \Gamma_L^{-1}(x) = \inf\{y > 0 : \Gamma_L^+(y) \leq x\}.$$

In order to simulate from (1) we need to be able to simulate from  $e^{-\lambda t} \int_0^{\lambda t} e^s dL_s$ . The key result here is the following infinite series representation of this type of integrals (Rosinski [21]):

**Proposition 5.2.** *Consider a subordinator  $L$  with positive increments. Let  $f$  be a positive and integrable function on  $[0, T]$ . Then*

$$(23) \quad \int_0^T f(s) dL_s = \sum_{i=1}^\infty \Gamma_L^{-1}(\alpha_i/T) f(Tr_i),$$

where the equality is understood in distributional sense,  $\{\alpha_i\}$  and  $\{r_i\}$  are two independent sequences of random variables such that  $r_i$  are independent copies of a uniform random variable in  $[0, 1]$  and  $\{\alpha_i\}$  is a strictly increasing sequence of arrival times of a Poisson process with intensity 1.

**Remark 5.3.** We note here that the convergence of the series (23) is often quite slow.

Using (23) we can then simulate a Lévy driven OU process. In particular, if  $\Delta$  denotes the time step, we will use the identity

$$\begin{aligned} Y_{t+\Delta} &= e^{-\lambda\Delta}(Y_t + e^{-\lambda t} \int_t^{t+\Delta} e^{\lambda s} dL_{\lambda s}) \\ (24) \quad &= e^{-\lambda\Delta}(Y_t + \int_0^\Delta e^{\lambda s} dL_{\lambda s}). \end{aligned}$$

Let us demonstrate the validity of our estimators modeling through the BDLP. We consider two cases: (a) when  $Y_0 \sim \text{Gamma}(a, b)$  and (b) when  $Y_0 \sim \text{IG}(a, b)$ , where IG stands for inverse Gaussian.

Regarding the  $\lambda$  parameter, we recall our estimators:  $\hat{\lambda}_{1,n} = -\frac{\log(\hat{\rho}_n(1))}{\Delta}$  and  $\hat{\lambda}_{2,n} = \arg\min_{\lambda} \sum_{h=1}^d (\hat{\rho}_n(h) - e^{-\lambda h \Delta})^2$ . In (10) we defined  $\hat{\lambda}_{1,n}$  and  $\hat{\lambda}_{2,n}$  for  $\Delta = 1$ , but of course one can generalize them to any  $\Delta > 0$ .

**5.1. Gamma OU model.** Assume that the driving Lévy process  $L$  is a compound Poisson process and in particular, that  $L_t = \sum_{n=1}^{N_t} x_n$  where  $N_t$  is Poisson with intensity parameter  $a$  and  $x_n$  are independent identically distributed  $\text{Gamma}(1, b)$  random variables. Using (20) we get that  $Y_0 \sim \text{Gamma}(a, b)$ . It is known (see [2]) that in this case

$$\Gamma_L^{-1}(x) = \max\{0, -\frac{1}{b} \log(\frac{x}{a})\}.$$

Using this and equations (23) and (24) we can easily simulate from a  $\text{Gamma}(a, b)$ -OU process. We also need to know how the parameters  $\mu$  and  $\sigma^2$  relate to  $a$  and  $b$ . Since  $\mathbb{E}L_1 = \mu$  and  $\text{Var}(L_1) = \sigma^2$  implies  $\mathbb{E}Y_0 = \mu$  and  $\text{Var}(Y_0) = \frac{\sigma^2}{2}$ , we have that  $a = 2\frac{\mu^2}{\sigma^2}$  and  $b = 2\frac{\mu}{\sigma^2}$ .

We simulated 100 independent paths of a gamma OU process of 1000 observations each, with time step  $\Delta = 0.1$ , using (24). We chose  $\mu = 2$ ,  $\sigma^2 = 0.25$  and in order to capture possible different behaviors of the intensity parameter we chose two different values for  $\lambda$ , 0.5 and 5.

Tables I and II, summarize the results for  $\theta_0 = (2, 0.25, 0.5)$  and for  $\theta_0 = (2, 0.25, 5)$  respectively.

True Values	Est. Values	Sample Std. Error	Comments
$\mu = 2$	1.995458	0.0702198	-
$\sigma^2 = 0.25$	0.2350207	0.05352894	-
$\lambda = 0.5$	0.566116	0.1126439	$\hat{\lambda}_n = \hat{\lambda}_{1,n}$
$\lambda = 0.5$	0.5879571	0.1441501	$\hat{\lambda}_n = \hat{\lambda}_{2,n}$

TABLE 1.  $\theta_0 = (2, 0.25, 0.5)$

True Values	Est. Values	Sample Std. Error	Comments
$\mu = 2$	2.003799	0.02094129	-
$\sigma^2 = 0.25$	0.2473567	0.01608991	-
$\lambda = 5$	5.12962	0.4463517	$\hat{\lambda}_n = \hat{\lambda}_{1,n}$
$\lambda = 5$	5.186585	0.5898125	$\hat{\lambda}_n = \hat{\lambda}_{2,n}$

TABLE 2.  $\theta_0 = (2, 0.25, 5)$ 

**5.2. Inverse Gaussian OU model.** It is well known that if  $Y_0 \sim \text{IG}(a, b)$ , then the Lévy density of  $Y$  is

$$(25) \quad \nu_Y(x) = \frac{1}{\sqrt{2\pi}} a x^{-3/2} e^{-\frac{1}{2} b^2 x}.$$

Consider now the Lambert-W function,  $L_w(\cdot)$ , which satisfies  $L_w(x)e^{L_w(x)} = x$ . As it is also shown in Gander and Stephens [11], equations (20), (21) and (22) imply that the inverse tail mass function of the BDLP of an IG(a,b)-OU process is given by

$$(26) \quad \Gamma_L^{-1}(x) = \frac{1}{b^2} L_w\left(\frac{a^2 b^2}{2\pi x^2}\right).$$

Using the latter and equations (23) and (24) we can easily simulate an IG(a,b)-OU process. We also need to know how the parameters  $\mu$  and  $\sigma^2$  relate to  $a$  and  $b$ . Since  $EL_1 = \mu$  and  $\text{Var}(L_1) = \sigma^2$  implies  $EY_0 = \mu$  and  $\text{Var}(Y_0) = \frac{\sigma^2}{2}$ , we have that  $a = \mu\sqrt{\frac{2\mu}{\sigma^2}}$  and  $b = \sqrt{\frac{2\mu}{\sigma^2}}$ .

We simulated 100 independent paths of an IG-OU process of 1000 observations each with time step  $\Delta = 0.1$ , using (24). As before, we chose  $\mu = 2$ ,  $\sigma^2 = 0.25$  and two different values for  $\lambda$ , 0.5 and 5.

Tables III and IV, summarize the results for  $\theta_0 = (2, 0.25, 0.5)$  and for  $\theta_0 = (2, 0.25, 5)$  respectively.

True Values	Est. Values	Sample Std. Error	Comments
$\mu = 2$	1.986862	0.06476202	-
$\sigma^2 = 0.25$	0.2331244	0.05235387	-
$\lambda = 0.5$	0.5581237	0.1128397	$\hat{\lambda}_n = \hat{\lambda}_{1,n}$
$\lambda = 0.5$	0.6050457	0.1376689	$\hat{\lambda}_n = \hat{\lambda}_{2,n}$

TABLE 3.  $\theta_0 = (2, 0.25, 0.5)$ TABLE 4.  $\theta_0 = (2, 0.25, 5)$ 

True Values	Est. Values	Sample Std. Error	Comments
$\mu = 2$	1.955288	0.03107831	-
$\sigma^2 = 0.25$	0.2452349	0.01750871	-
$\lambda = 5$	5.05211	0.4262788	$\hat{\lambda}_n = \hat{\lambda}_{1,n}$
$\lambda = 5$	5.158421	0.659508	$\hat{\lambda}_n = \hat{\lambda}_{2,n}$

## 6. REAL DATA ANALYSIS

In 1993, the Chicago Board Options Exchange (CBOE) introduced the CBOE volatility index, VIX, and it quickly became a popular measure for stock market volatility. In 2003, the VIX methodology was updated (see [www.cboe.com](http://www.cboe.com) for more details on the old and new VIX methodology). VIX measures the implied volatility of S&P 500 index options and it provides a minute-by-minute snapshot of the markets expectancy of volatility over the next 30 calendar days.

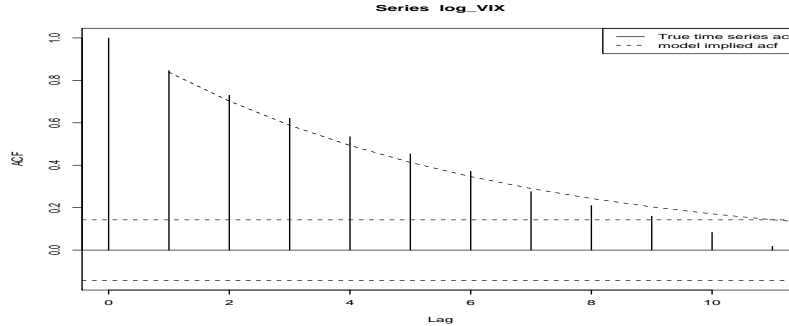
We fitted the gamma OU model and the IG OU model to daily log opening values of the VIX for the year 2004 (VIX values are calculated using the new methodology). The data are taken from [www.cboe.com](http://www.cboe.com). We use the values from 1/2/2004 till 9/30/2004 for the calibration of the model (in total 189 data points) and the values from 10/1/2004 till 11/30/2004 (in total 41 data points) for testing the model.

Table V summarizes the estimators, given by (10), of the parameters of the model. We used  $\hat{\lambda}_{2,n}$  to estimate  $\lambda$ .

parameter	$\mu$	$\sigma^2$	$\lambda$
estimated value	2.781769	0.01919740	0.1767250

TABLE 5. Estimated values for the parameters.

In Figure 1, we see the first 10 lags of the empirical autocorrelation function of the  $\log(\text{VIX})$  for 1/2/2004 till 9/30/2004 versus the theoretical autocorrelation function of the OU model with  $\lambda = 0.1767250$ , i.e.  $\rho(h) = e^{-0.1767250h}$ .

FIGURE 1. True time series acf versus the model implied acf with  $\lambda = 0.1767250$ .

As we saw before, the autocorrelation function of an OU model is exponentially decreasing, i.e. it has the form  $e^{-\lambda h}$ . Figure 1 shows that  $e^{-0.1767250h}$  approximates sufficiently well the empirical autocorrelation function of  $\log(\text{VIX})$  for 1/2/2004 till 9/30/2004, which is also exponentially decreasing. Hence, we conclude that an OU model is a good candidate for describing this data set.

To investigate the model fit, we performed a Ljung-Box test for the squared residuals. We used the estimated values from Table V and since our data is daily opening values we chose  $\Delta = 1$  for the time step. The test statistic used 10 lags

of the empirical autocorrelation function. The gamma OU model performed better than the IG-OU model. The null hypothesis was not rejected at the 0.05 level and the  $p$ -value was quite high, 0.6165. In Figure 2 we see the empirical autocorrelation function of the residuals of  $\log(\text{VIX})$  and in Figure 3 we see the actual residuals of the gamma OU model.

In Figure 4 we see in one figure: the actual time series from 10/1/2004 till 11/30/2004, the one step ahead predicted time series and 95% bootstrap upper and lower confidence bounds of the one step ahead predicted time series. In order to create the one step ahead predicted time series we averaged over 50 paths. We observe that the real time series (solid line) is most of the time within the 95% bootstrap upper and lower confidence bounds of the one step ahead predicted time series (dotted lines), with very few exceptions.

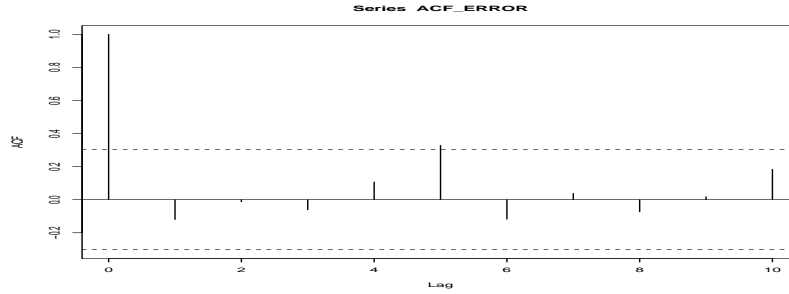


FIGURE 2. Empirical acf for the residuals of  $\log(\text{VIX})$ .

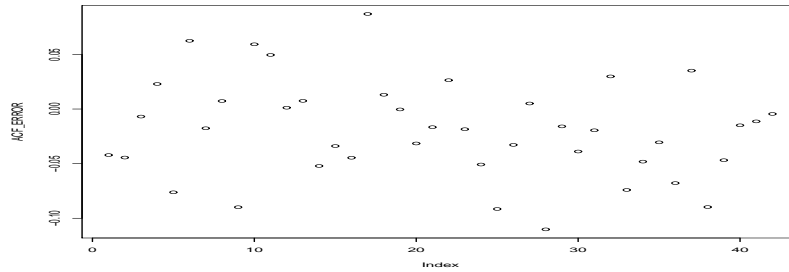


FIGURE 3. The actual residuals.

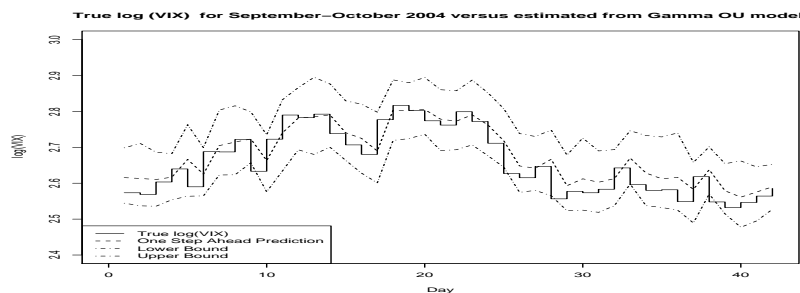


FIGURE 4. Actual time series versus one step ahead predicted time series.

## 7. DISCUSSION AND FUTURE WORK

In this paper, we consider an Ornstein-Uhlenbeck process driven by a general Lévy process. We derive strong consistent estimators for the parameters of the model and we prove that they are asymptotically normal. Using simulated data, we show that the estimators perform well at least for a gamma OU model and for an IG-OU model. Lastly we fit the model to real data and we see that a Lévy driven OU model is a good candidate for describing  $\log(\text{VIX})$ .

There are some interesting extensions to the model studied in this paper. One such extension is a coupled two dimensional OU process driven by a two dimensional Lévy process. This model is important for financial applications, since it could be used to model log of the price and stochastic volatility simultaneously. An interesting question for financial applications is option pricing in these type of models (see Nicolato and Venardos [20] for some recent related results). These questions will be addressed in future work.

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